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INSTABILITY OF THE DISPLACEMENT FRONTS OF NON-NEWTONIAN FLUIDS IN A HELE–SHAW CELL†

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The destabilizing effect of the shear-thinning of a non-Newtonian fluid and its elasticity in the plane-parallel and circular displacement fronts in a Hele–Shaw cell is investigated, as well as the influence of the elasticity of a fluid on the shape of the Saffman–Taylor "fingers" which are formed when primary instability develops in the fronts. The treatment is carried out separately for a pure viscous non-Newtonian fluid and for a visco-elastic fluid (with Newtonian viscous behaviour under steady shear flow). As a result of the multistage development of instability in the displacement of one fluid by another in a Hele–Shaw cell (as the flow in the narrow gap between large plane parallel plates is traditionally called), various structures can be formed from an individual displacement finger up to complex patterns of a fractal nature. The form of these structures and the conditions for their formation are found to be sensitive to the rheological characteristics of the fluids. © 1997 Elsevier Science Ltd. All rights reserved.

In Section 1, we consider the linear stability of rectilinear and circular displacement fronts for a non-Newtonian, non-linear viscous fluid which is displaced by another fluid, taking account of surface tension in the formulation of the plane problem which corresponds to the averaged motion over the gap width. It is shown that the critical wavelength of unstable small perturbations for a rectilinear front and the critical radius for the onset of instability in the case of radial displacement in pseudoplastic fluids is less than in viscous Newtonian fluids. Unstable perturbations also grow more rapidly in pseudoplastic fluids. These general conclusions are more easily achieved when there is a power dependence of the viscosity on the shear stress. They are true regardless of some uncertainty in the methods for comparing power-law and Newtonian viscous fluids.

In Section 2, we estimate the similar effect of the elasticity of the visco-elastic fluids which are being displaced on the development of instability in plane-parallel and circular fronts and on the advance of a finger of an inviscid fluid in a Hele–Shaw cell. An acceleration in the growth of perturbations was established earlier in [1] by a numerical analysis of the equations of small perturbations of a plane-parallel displacement front of a viscoelastic fluid with the Oldroyd-B model. The shear flow in the gap between the parallel plates of the cell is characterized, in the case of this model, by the absence of an anomalous viscosity effect, and the effective viscosity remains constant. This result is thereby indicative of the destabilizing effect of elastic normal stresses. The complexity of the analysis in [1] is due to the fact that the normal stresses were taken into account both in the boundary conditions on the displacement front as well as in the determining differential equations. It will be shown that, at least in fluids with low elasticity, the effect of the normal stresses in the boundary condition on the front is a prominent factor in accelerating the growth of small perturbations of the displacement front. It is assumed that, locally within the part of the cell filled with the fluid, the flow obeys the usual Boussinesq relation for a viscous fluid, which is equivalent to the Darcy law in the case of a porous medium. Results are then obtained quite simply in an analytic form both for a rectilinear and a circular displacement front. A uniformly moving finger of the inviscid fluid which is being displaced is formed during the subsequent development of unstable perturbations. When the effect of the low elasticity of the fluid which is being displaced solely on the front is taken into account as usual, it is shown that such a finger becomes less sharp due to the elasticity.

1. DISPLACEMENT FRONTS OF NON-LINEAR VISCOUS FLUIDS

The modified Darcy law for the flow of a non-linear viscous fluid. We consider the flow of an incompressible, non-Newtonian fluid (with a general form of the dependence of the coefficient of viscosity η on

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the second invariant of the rate of deformation tensor and $v_z \approx 0$, $\partial/\partial z \gg \partial/\partial x, \partial/\partial y$) in the gap $-h \leqslant z \leqslant h$ between two closely spaced smooth parallel plates (Hele–Shaw flow).

In the inertialess approximation of a thin layer (the assumption of the smallness of the Reynolds numbers is natural in the case of a narrow gap), the equations of motion reduce to a relation for the balancing of the pressure forces and viscosity and, together with the incompressibility condition and the defining equation, they take the form

$$\frac{\partial p}{\partial z}=0,\quad \nabla p=\frac{\partial \boldsymbol{\tau}}{\partial z},\quad \nabla\cdot\mathbf{v}=0,\quad \boldsymbol{\tau}=\eta\left(\left|\frac{\partial \mathbf{v}}{\partial z}\right|\right)\frac{\partial \mathbf{v}}{\partial z},\quad \nabla\equiv(\partial/\partial x,\ \partial/\partial y) \tag{1.1}$$

The description of the flow becomes actually two-dimensional: $\mathbf{v} = (v_x, v_y, 0)$ is the velocity vector and $\boldsymbol{\tau} = (\tau_x, \tau_y)$ is the two-dimensional "shear stress vector". In general, it is possible to drop the third coordinate z in the equations obtained by integrating with respect to it. The first integration of the equations of motion, taking account of the fact that there are no shear stresses in the meridian plane $z = 0$ (in the plane of symmetry of the flow), after inversion of the defining relation, gives

$$\frac{\partial \mathbf{v}}{\partial z}=g(\tau)\frac{\boldsymbol{\tau}}{\tau},\quad \left|\frac{\partial \mathbf{v}}{\partial z}\right|=g(\tau),\quad \boldsymbol{\tau}=z\nabla p$$

and, on carrying out a second integration with respect to z taking account of the no-slip condition for the fluid on the solid walls and averaging over the whole thickness of the gap between the plates, we arrive at the relations for the averaged flow velocity through a Hele–Shaw cell

$$\mathbf{u}=-M\nabla p,\quad \nabla\cdot\mathbf{u}=0,\quad \mathbf{u}\equiv\langle\mathbf{v}\rangle=\frac{1}{2h}\int_{-h}^{h}\mathbf{v}\,dz$$
$$M=M(\tau_w)=\frac{h^2}{\tau_w^3}\int_0^{\tau_w}\tau g(\tau)d\tau,\quad \tau_w\equiv\tau|_{z=\pm h}=h|\nabla p| \tag{1.2}$$

By virtue of the well-known analogy with flow in a porous medium, these relations, for brevity, are subsequently called the modified (non-linear) Darcy law. In the case of a viscous Newtonian fluid, the defining equation is linear ($g(\tau) = \tau/\eta_0$), and the "Darcy law" turns out to be linear with a constant mobility $M = h^2/(3\eta_0)$. A non-Newtonian fluid is non-linear and the mobility M depends on the magnitude of the shear stresses on the walls τ_w which is proportional to the magnitude of the pressure gradient. For example, in the case of a power-law fluid

$$g(\tau)=\left(\frac{\tau}{K}\right)^{1/n},\quad M=\frac{nh^2}{2n+1}\frac{g(\tau_w)}{\tau_w} \tag{1.3}$$

Rectilinear front of the displacement of one fluid by another and its stability. We now consider the problem of the homogeneous displacement of a non-linear viscous fluid with a constant velocity U within the framework of an averaged plane description with Eqs (1.2). It has the simple solution

$$u_{i1}=U\delta_{ix},\quad p_1=p_0-\frac{\alpha}{h}+\frac{\partial p_1}{\partial x}(x-Ut),\quad x>Ut$$

$$u_{i2}=U\delta_{ix},\quad p_2=p_0+\frac{\partial p_2}{\partial x}(x-Ut),\quad x<Ut$$

and, by virtue of the modified Darcy law, the constant values of the pressure gradient are linked with the constant rate of displacement by the relations

$$U=-M_1\left(h\left|\frac{\partial p_1}{\partial x}\right|\right)\frac{\partial p_1}{\partial x}=-M_2\left(h\left|\frac{\partial p_2}{\partial x}\right|\right)\frac{\partial p_2}{\partial x} \tag{1.4}$$

We label the characteristics of the fluid being displaced and the displacing fluid with subscripts 1 and 2 respectively. Here, the difference in the mobilities M_1 and M_2 is attributable to the difference in their constitutive equations.

On the common boundary of separation (the displacement front) $x = Ut$, the velocities of the flows are identical and the pressure undergoes a discontinuity of magnitude α/h due to capillarity (the interphase boundary, which is rectilinear in the (x, y) plane, is strongly curved in the third direction and its curvature is inversely proportional to the gap width). The expression used for the magnitude of the capillary jump corresponds to the case of an ideally wetting fluid which is displaced; however, it only changes quantitatively if the wetting is non-ideal but the contact angle remains constant regardless of the motion of the displacement front.

Small distortions of the displacement front in the plane of the flow

$$x = Ut + \xi(y, t) \tag{1.5}$$

give rise to small perturbations in the velocity and pressure fields of the fluids. The components of the velocity perturbations in the direction of the normal to the front are defined by the linearized boundary condition

$$\frac{\partial \xi}{\partial t} = u'_{x1}\big|_{x=Ut} = u'_{x2}\big|_{x=Ut} \tag{1.6}$$

In the linear approximation, the modified Darcy law reduces to linear relations between the velocity and pressure perturbations

$$u'_{xs} = -L_s \frac{\partial p'_s}{\partial x}, \quad u'_{ys} = -M_s \frac{\partial p'_s}{\partial y}, \quad s = 1,2$$
$$M_s \equiv M_s(\tau_{ws}) > 0, \quad L_s \equiv \frac{h^2}{\tau_{ws}} g_s(\tau_s) - 2M_s > 0 \tag{1.7}$$

and, by virtue of the incompressibility condition, the pressure perturbations satisfy the (elliptic type) equations

$$L_s \frac{\partial^2 p'_s}{\partial x^2} + M_s \frac{\partial^2 p'_s}{\partial y^2} = 0$$

Since the equations for the perturbations are linear and have constant coefficients (because of the homogeneity of the perturbed flow) it is convenient to consider elementary perturbations with an exponential dependence on y and t. Finally, we have

$$\xi = ae^{i\theta}, \quad \theta \equiv ky - \omega t$$
$$\left(u'_{xs},\ u'_{ys},\ p'_s\right) = -\omega a\left(i,\ \frac{k}{|k|}\left(\frac{M_s}{L_s}\right)^{1/2},\ \frac{i}{|k|}\left(L_s M_s\right)^{-1/2}\right)\exp(i\theta + \psi_s)$$
$$\psi_{1,2} = \mp|k|\left(\frac{M_{1,2}}{L_{1,2}}\right)^{1/2}(x - Ut)$$

If we again take into account the linearized boundary condition for the pressure jump due to the action of surface tension when there are perturbations of the front curvature in the plane of the flow (this constraint on the distortions of the front solely in the (x,y) plane appears to be somewhat artificial. However, as the never-ending discussion in the case of Newtonian fluids [2] shows, it is entitled to exist as a certain first approximation), that is

$$\alpha \frac{\partial^2 \xi}{\partial y^2} = \left(\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}\right)\xi + \left(p'_1 - p'_2\right)\big|_{x=Ut}$$

then we obtain the dispersion equation

$$-i\omega = |k|\left(\frac{\partial p_2}{\partial x} - \frac{\partial p_1}{\partial x} - \alpha k^2\right)\left[(L_1M_1)^{-1/2} + (L_2M_2)^{-1/2}\right]^{-1} \tag{1.8}$$

which relates $-i\omega$ to $|k|$ and the defining instability condition and the rate of exponential growth of the small perturbations.

The surface tension at the interface between two fluids leads to the appearance of a critical (minimum) value of the wavelength of the growing perturbations

$$\lambda_c = 2\pi\alpha^{1/2}\left(\left|\frac{\partial p_1}{\partial x}\right| - \left|\frac{\partial p_2}{\partial x}\right|\right)^{-1/2} \tag{1.9}$$

Only perturbations with $\lambda > \lambda_c$ are unstable and perturbations with a wavelength $\lambda_c\sqrt{3}$ grow most rapidly. A rectilinear displacement front is always unstable with respect to much longer wavelength perturbations when $|\partial p_1/\partial x| > |\partial p_2/\partial x|$ regardless of the magnitude of the surface tension.

At first glance, result (1.8) indicates, at least, that the instability criterion is independent of the type of fluid if the pressure gradients ahead of the front and behind it are equalized for the various constitutive relations of the fluids (for different pairs of fluids). However, in the case of a selected pair of fluids, it is not possible, generally speaking, to specify the pressure gradients in them independently as is seen from (1.4). The different, but matching, pressure gradients must ensure one and the same rate of steady advance of the front when the fluids have different mobilities. The rate of growth of unstable perturbations turns out to be even more sensitive to the characteristics of the fluids.

In the case of a power-law fluid, it follows from (1.4) and (1.7) that

$$M_s = n_s L_s = \frac{n_s h^{1+1/n_s}}{(2n_s+1)K_s^{1/n_s}}\left|\frac{\partial p_s}{\partial x}\right|^{1/n_s - 1} = U\left|\frac{\partial p_s}{\partial x}\right|^{-1} \tag{1.10}$$

and dispersion equation (1.8) reduces to the formula

$$-i\omega = U|k|\left(\left|\frac{\partial p_1}{\partial x}\right| - \left|\frac{\partial p_2}{\partial x}\right| - \alpha k^2\right)\left(\left|\frac{\partial p_1}{\partial x}\right|\sqrt{n_1} + \left|\frac{\partial p_2}{\partial x}\right|\sqrt{n_2}\right)^{-1} \tag{1.11}$$

with an obvious dependence of the rate of growth of the unstable perturbations on the power-law index n.

In the case of a specified rate of advance of the front U, the parametrization can be changed using relations (1.4). In the case of a power-law fluid, inversion of (1.4) gives (see also (1.10))

$$\left|\frac{\partial p_s}{\partial x}\right| = \frac{K_s}{h}\left[\left(2 + \frac{1}{n_s}\right)\frac{U}{h}\right]^{n_s} = \eta_{ws}\left(2 + \frac{1}{n_s}\right)\frac{U}{h}$$

In the last equality, the viscosities of the two fluids on the plates which bound the flow have been introduced in accordance with the definition $\eta_w = \tau_w/g(\tau_w)$. These viscosities are completely determined by the magnitude of U. When this is taken into account, formula (1.1) can be rewritten in the somewhat different form

$$-i\omega = U|k|\frac{U[\eta_{w1}(2+1/n_1) - \eta_{w2}(2+1/n_2)] - \alpha k^2 h^2}{\eta_{w1}(2+1/n_1)\sqrt{n_1} + \eta_{w2}(2+1/n_2)\sqrt{n_2}}$$

In the special case of Newtonian fluids ($n_1 = n_2 = 1$), the result is identical with the classical result from [3, 4].

We shall now discuss the last formula for the case of a displacing fluid with such a small viscosity that it may be neglected by putting $\eta_{w2} = 0$. Then, for the critical wavelength which separates stable and unstable perturbations and for the rate of growth of the unstable perturbations, we shall have

$$\lambda_c = 2\pi h\left(\frac{\alpha}{U\eta_w}\,\frac{n}{2n+1}\right)^{1/2}, \quad -i\omega = U^2|k|\left[1 - \left(\frac{k\lambda_c}{2\pi}\right)^2\right]\frac{1}{\sqrt{n}}$$

It is clear from this that, in the case of pseudoplastic fluids ($n < 1$), the critical wavelength is smaller than in the case of a Newtonian fluid with a viscosity which is equal to the effective viscosity of the fluid which is displaced at the same rate of displacement. The rate of growth of unstable perturbations (of fixed wavelength) is higher in the case of a non-Newtonian fluid. The opposite assertions apply in the case of dilatant fluids ($n > 1$).

We further note the possibility of quantitative variant readings when comparing the results for Newtonian and non-Newtonian fluids due to the variability of the viscosity of the latter. For instance, instead of the characteristic of the integral slope of the flow curve and the viscosity $\eta_w = \tau_w/g(\tau_w)$, it is possible to choose the characteristic of the local slope of the flow curve and the "tangential viscosity" $\bar{\eta}_w = (\partial g/\partial \tau_w)^{-1}$. In the case of a power-law fluid, they differ by a factor which is equal to the power-law index $\bar{\eta}_w = n\eta_w$. It is therefore clear that, when the viscosity of a Newtonian fluid is correlated with the "tangential viscosity" of a non-Newtonian fluid, the preceding conclusion concerning the drop in the critical wavelength and the more rapid growth of unstable perturbations is only reinforced.

Development of instability in the radial displacement of a non-Newtonian fluid. In the case of purely radial flow of an incompressible fluid, the velocity (averaged over the gap width) is inversely proportional to the radius. The pumping in of a displacing fluid (with subscript 2) at the origin of the system of coordinates with a bulk flow rate per unit time $Q(t)$ brings about a radial flow of both fluids with a velocity

$$u = U\frac{R}{r}, \quad u = u_{r1}, \quad r > R; \quad u = u_{r2}, \quad r < R; \quad U = \frac{dR}{dt} = \frac{Q}{4\pi hR}$$

Pressure gradients which are defined by the implicit relation

$$U\frac{R}{r} = -M_s\left(h\left|\frac{\partial p_s}{\partial r}\right|\right)\frac{\partial p_s}{\partial r}, \quad s = 1, \quad r < R(t), \quad s = 2, \quad r > R(t) \tag{1.12}$$

correspond to this velocity field according to the modified Darcy law (1.2). Here, unlike in the preceding case of rectilinear displacement, the flow is not homogeneous and the pressure gradients (and mobilities M_s) are constant.

Small perturbations of the circular displacement front

$$r = R(t) + \xi_m(t)e^{im\varphi}, \quad \xi \ll R \tag{1.13}$$

induce perturbations in the velocity and pressure fields of the fluids. The perturbations of the angular components

$$u'_{\varphi s} = \frac{i}{m}A_s(t)\frac{d}{d\rho}(\rho B_s)e^{im\varphi} \tag{1.14}$$

correspond, according to the incompressibility condition, to perturbations of the radial component of the flow velocity of the form

$$u'_{rs} = A_s(t)B_s(\rho)e^{im\varphi}, \quad \rho \equiv r/R(t) \tag{1.15}$$

Note that, in the general case, perturbations of the velocity field do not admit of representations of the form (1.14), (1.15) with a splitting of the dependences on the radial and time variables. We therefore restrict ourselves to special classes of flows which justify a posteriori the assumptions which have been adopted. The linearized boundary condition for the equality of the normal components of the velocities in the perturbed displacement front and the front velocities

$$\left.\left(\frac{\partial u_{rs}}{\partial r}\xi_m e^{im\varphi} + u'_{rs}\right)\right|_{r=R} = \frac{d\xi_m}{dt}e^{im\varphi}$$

enable us to express the amplitude of the velocity perturbations $A_s(t)$ in terms of the amplitude of the perturbations of the front $\xi_m(t)$

$$A_1(t) = A_2(t) = A(t) \equiv \frac{1}{R}\frac{d}{dt}(\xi_m R), \quad B_1(1) = B_2(1) = 1$$

On linearizing the modified Darcy law, for small perturbations of the velocities and pressures, we obtain the relations

$$\begin{gathered} u'_{rs} = -L_s \frac{\partial p'_s}{\partial r}, \quad u'_{\varphi s} = -M_s \frac{1}{r}\frac{\partial p'_s}{\partial \varphi} \\ M_s \equiv M_s\left(h\left|\frac{\partial p_s}{\partial r}\right|\right), \quad L_s \equiv h\left|\frac{\partial p_s}{\partial r}\right|^{-1} g_s\left(h\left|\frac{\partial p_s}{\partial r}\right|\right) - M_s \end{gathered} \tag{1.16}$$

which enable us to express pressure perturbations in terms of the functions $B_s(\rho)$ and to obtain the differential equations for these functions

$$\begin{gathered} p'_s = -\frac{1}{m^2} R(t)A(t)\frac{\rho}{M_s}\frac{d}{d\rho}[\rho B_s(\rho)]e^{im\varphi} \\ L_s \frac{d}{d\rho}\left[\frac{\rho}{M_s}\frac{d}{d\rho}\rho B_s(\rho)\right] = m^2 B_s(\rho) \end{gathered} \tag{1.17}$$

In the problem being considered concerning the perturbations of a circular displacement front, the coefficient functions M_s and L_s depend, generally speaking, on the radial coordinate and the time, that is, on ρ and t. This, as has already been stated, contradicts the assumption that B_s depends solely on ρ. However, these assumptions are certainly satisfied for certain classes of flows. For instance, under displacement conditions with a constant rate of advance of the front, $U = U_0$, which ensures a linearly increasing fluid flow rate $Q(t) = 4\pi h U_0^2 t$, these equations make sense in the cases of arbitrary non-Newtonian fluids. Then, the velocities of the fluids depend solely on ρ ($v_{rs} = U_0/\rho$) and, by virtue of (1.12) and (1.16), this is also true with respect to the pressure gradients $\partial p_s/\partial r$, the mobilities M_s and the coefficient functions L_s. In the case of power-law fluids, Eq. (1.17) makes senses regardless of the displacement conditions

$$\begin{gathered} M_s = n_s L_s = \frac{n_s h^2}{(2n_s+1)K_s}\left[\frac{2n_s+1}{h\rho n_s}U(t)\right]^{1-n_s} \\ \rho^{n_s-1}\frac{d}{d\rho}\left[\rho^{2-n_s}\frac{d}{d\rho}(\rho B_s(\rho))\right] = n_s m^2 B_s(\rho) \end{gathered}$$

and allows of the power solutions

$$\begin{gathered} B_s(\rho) = \rho^{\nu_s} \\ 2(\nu_1+1) = -(1-n_1) - \sqrt{(1-n_1)^2 + 4n_1 m^2} \\ 2(\nu_2+1) = -(1-n_2) + \sqrt{(1-n_2)^2 + 4n_2 m^2} \end{gathered}$$

Here, the choice of the signs in front of the square roots is dictated by the requirement that the pressure perturbations ($p'_s \sim \rho^{2+\nu_s-n_s}$) should decrease on moving away from the front in the external fluid and their regularity close to the source at the origin of the system of coordinates in the internal fluid. In addition, the velocity perturbations ($v'_{rs} \sim v'_{\varphi s} \sim \rho^{\nu_s}$) decay in the displaced fluid far from the front and are less singular than the main flow close to the source of the displacing fluid.

Finally, the linearized boundary condition for the pressure jump in the perturbed front due to capillary pressure and written under the assumption that the meniscus is not perturbed in a direction perpendicular to the plates and neglecting the relatively small normal viscous stresses

$$\left[\left(\frac{\partial p_1}{\partial r}-\frac{\partial p_2}{\partial r}\right)\xi_m e^{im\varphi}+\left(p_1'-p_2'\right)\right]_{r=R}=\alpha\frac{1-m^2}{R^2}\xi_m e^{im\varphi}$$

leads to an equation for the amplitude of the perturbations of a circular front

$$\frac{d\xi_m}{dt}R\left(\frac{b_1}{M_1}-\frac{b_2}{M_2}\right)_{r=R}+\xi_m\frac{dR}{dt}\left(\frac{b_1+m^2}{M_1}-\frac{b_2+m^2}{M_2}\right)_{r=R}=$$
$$=m^2(m^2-1)\frac{\alpha}{R^2}\xi_m,\quad b_s=\left.\frac{d\rho B_s}{d\rho}\right|_{\rho=1} \tag{1.18}$$

The mobilities M_s in the neighbourhood of the front ($r\to R$) are solely functions of its velocity and the resulting equation which is linear with respect to ξ_m enables one to determine the rate of growth of the distortions of the front $d\ln\xi_m/dt$.

We shall further restrict the treatment to the analysis of the case when a power-law fluid is displaced by a fluid with a negligibly small viscosity (the subscript s is then superfluous). In the case of a power-law fluid

$$M|_{r=R}=\frac{h}{K}\left(\frac{nh}{2n+1}\right)^n\left(\frac{dR}{dt}\right)^{1-n}$$
$$b=\frac{d}{d\rho}(\rho B)|_{\rho=1}=1+\nu=-\frac{1}{2}\left[1-n+\sqrt{(1-n)^2+4nm^2}\right]$$

and the equation for the rate of growth of the amplitude of the perturbations (1.18) is simplified

$$\frac{1}{\xi_m}\frac{d\xi_m}{dt}=-\frac{1}{R}\frac{dR}{dt}\frac{1+\nu+m^2}{1+\nu}+\frac{m^2(m^2-1)\alpha h}{(1+\nu)K}\left(\frac{nh}{2n+1}\right)^n\frac{1}{R^3}\left(\frac{dR}{dt}\right)^{1-n} \tag{1.19}$$

The case of perturbations with a lower wave number $m = 1$ is the simplest here. The amplitude equation degenerates into the trivial equation $d\xi_1/dt = 0$. Neutral perturbations require, generally speaking, additional analysis in the next order of perturbation theory (or that the low viscosity of the displacing fluid which was neglected is taken into account). However, they will not be discussed further here as they are not associated with distortions of a circular front and only bring about its displacement as a whole.

For perturbations with $m > 1$, we have $1 + \nu + m^2 > 0$ and, for an exponential change in the flow rate of the displacing fluid with time (or when the radius of the front increases)

$$Q=qt^\gamma,\quad R=\sqrt{\frac{q}{2\pi h(1+\gamma)}}t^{(1+\gamma)/2}$$
$$U=\frac{dR}{dt}=\phi R^{(\gamma-1)/(\gamma+1)},\quad \phi\equiv\frac{1+\gamma}{2}\left[\frac{q}{2\pi h(1+\gamma)}\right]^{1/(1+\gamma)}$$

the equation for the rate of growth of the perturbations (1.19) can be rewritten in a more obvious form

$$\frac{1}{\xi_m}\frac{d\xi_m}{dR}=\frac{\mu}{R}\left[1-\left(\frac{R_c}{R}\right)^\sigma\right] \tag{1.20}$$
$$\mu\equiv\frac{1+\nu+m^2}{1+\nu}>0,\quad \sigma\equiv 2-n\frac{1-\gamma}{1+\gamma},\quad R_c^\sigma=\frac{m^2(m^2-1)\alpha h}{(1+\nu+m^2)K}\left[\frac{nh}{(2n+1)\phi}\right]^n$$

Here R_c is the critical radius of the circular displacement front, and, when there is an increase in this in the case when $\sigma > 0$, the derivative $d\xi_m/dt$ becomes positive and instability starts to develop (the opposite holds when $\sigma < 0$). The existence of a critical radius is associated with the competition between the instability of a viscous flow and the stabilizing action of the surface tension, so that it is completely equivalent to the existence of a critical wavelength λ_c in the problem in the case of a plane displacement front. According to (1.20), the condition $\sigma > 0$ for a constant flow rate of the displacing fluid ($\gamma = 0$) implies a constraint on the power-law index $n < 2$, and it is only in the case of the uniform and accelerated expansion of a circular front ($\gamma \geqslant 1$) that it is satisfied without any such constraints.

Equation (1.20) for the rate of growth of perturbations (in the radial variable R it can be considered as in an "extended" time variable) is easily integrated

$$\xi_m = \xi_{m0}\left(\frac{R}{R_c}\right)^{\mu} \exp\left[\frac{\mu}{\sigma}\left(\frac{R_c}{R}\right)^{\sigma}\right] \tag{1.21}$$

In the case of a large expansion of the front, $R \gg R_c$, the growth in the perturbations becomes of the power type (rather than exponential as in the case of a straight front) and the exponent μ increases as the exponent n of the constitutive law for the fluid decreases

$$\frac{\partial\mu}{\partial n} = -\frac{m^2(1+\nu+m^2)}{(1+\nu)^2\sqrt{(1-n)^2+4nm^2}} < 0, \quad \mu\big|_{n=1} = m-1$$

so that in the case of fronts with a large radius, the development of instability in a pseudoplastic fluid ($n < 1$) must occur more rapidly than in a Newtonian (linearly viscous) fluid while the opposite effect occurs in the case of a dilatant fluid ($n > 1$). The effect on the exponential function in (1.21), which is important in the case of the almost critical development of perturbations, may be the opposite and, in fact, $\partial\sigma/\partial n < 0$ when $\gamma < 1$.

Instability of short wavelength angular harmonics sets in later than in the case of harmonics with smaller m. The critical radius increases monotonically as the number m increases, if $\sigma > 0$

$$\left[R_{c,m} / R_{c,m+1}\right]^{\sigma} < 1$$

The parametric dependence of the instability characteristics is obtained most simply for a uniformly expanding front ($\gamma = 1$). In general, the magnitude of σ then turns out to be independent of the properties of the displaced fluid ($\sigma = 2$), and we have

$$R = U_0 t, \quad Q = q_0 t, \quad q_0 = 4\pi h U_0^2 = \text{const}$$

$$\xi_m = \xi_{m0}\left(\frac{R}{R_c}\right)^{\mu} \exp\left[\frac{\mu}{2}\left(\frac{R_c}{R}\right)^{2}\right], \quad R_c^2 = \frac{m^2(m^2-1)\alpha h}{(1+\nu+m^2)K}\left[\frac{nh}{(2n+1)U_0}\right]^{n}$$

In order to compare the critical radii for fluids with different viscosity dependences, it should be specified which values of the viscosities, in fact, were recorded. In the case of radial flow in a Hele–Shaw cell, the shear stresses on its walls depend on the radial coordinate and, for this reason, a non-linear viscosity turns out to be variable even on the walls

$$\eta_w \equiv \frac{\tau_w}{g(\tau_w)} = K\left[\frac{nh}{(2n+1)U_0}\,\frac{r}{R}\right]^{1-n}$$

It is convenient here to record viscosities close to the front $\eta_{wR} \equiv \eta_w|_{r=R}$, and when this is done we obtain for the critical radius

$$R_c^2 = \frac{m^2(m^2-1)n}{(1+\nu+m^2)(2n+1)}\,\frac{\alpha h^2}{U_0\eta_{wR}}, \quad \frac{\partial R_c}{\partial n} > 0$$

Hence, the critical radius of a uniformly propagating displacement front of a pseudoplastic fluid is smaller than in the case of the displacement of a Newtonian fluid. Note that, when a comparison is made using the "tangential" viscosity close to the front $\bar{\eta}_{wR} = (\partial g/\partial \tau_{wR})^{-1} = n\eta_{wR}$, the derivative $\partial R_c/\partial n$ is found to be still greater.

Finally, when surface tension is neglected, the displacement of a low-viscosity gas by an arbitrary non-linear viscous liquid from a Hele–Shaw cell turns out to be unstable with respect to perturbations of any wavelength. This applies to straight and circular displacement fronts in equal measure.

Surface tension makes short waves in a rectilinear uniformly moving front stable. In addition, the critical wavelength (which separates stable and unstable waves) decreases in shear thinning fluids (pseudoplastic fluids) and increases in dilatant fluids compared with the case of a viscous fluid at equal viscosities close to the displacement front. The rate of exponential growth of small unstable perturbations (with a wavelength greater than the critical wavelength) is greater in pseudoplastic fluids.

Under the action of surface tension, the growth of a circular front is stabilized up to a certain critical radius. On further expansion, the lower angular harmonics initially become unstable and then the higher angular harmonics (this all occurs within the framework of the linearized description). Here, even the growth in small perturbations is not exponential, but, in the case of large radius fronts, is close to a power-law relation. Under conditions where there is a uniform expansion of the front, the critical radius is smaller and the rate of growth of unstable perturbations is greater in a pseudoplastic fluid than in a viscous Newtonian fluid.

We now arrive at a general conclusion concerning the lower stability of the process of the displacement of a shear thinning fluid.

Instability in the displacement of non-linear viscous fluids has been analysed previously in [1, 5–9]. The modified Darcy law and instability criteria of the form of (1.8) with a pressure gradient parametrization in the case of a plane displacement front in a porous medium (that is, in a problem which, in many respects, is similar to the problem of the flow in a Hele–Shaw cell) has been considered in [5], where earlier papers are also mentioned. In the case of fluids which follows a power law, instability in rectilinear and circular fronts in a porous medium has been considered in [6–8]. In addition, a circular front has been discussed [7, 8] without taking account of such an important factor as the surface tension. The analysis in [1] was confined to a plane front in a Hele–Shaw cell (with not the most successful parametrization). The problem of the stability of a circular front in a Hele–Shaw cell has been discussed in [9] in reasonable detail but, in this case, an error was made in formulating the boundary condition for the velocities on the front which was reflected in the results obtained. Note that, in [1, 9], an exceedingly large spatial description was used with a subsequent averaging of the results over the thickness of the narrow gap. Finally, invariance of the critical radius for angular harmonics with $m = 2$ when the exponent of the rheological relation for a non-Newtonian fluid changes was unjustifiably assumed in [9] in the qualitative interpretation of the results.

2. DISPLACEMENT FRONTS OF VISCO-ELASTIC FLUIDS

Development of the instability of a plane-parallel displacement front. We now consider a steady pressurized shear flow in a Hele–Shaw cell in the case of a visco-elastic fluid is modeled with the Jeffreys–Oldroyd-B constitutive equations

$$\theta_1 \frac{\delta}{\delta t}\left(\sigma_{ij} - 2\eta\frac{\theta_1}{\theta_2}e_{ij}\right) + \sigma_{ij} = 2\eta e_{ij}$$

$$\frac{\delta\sigma_{ij}}{\delta t} \equiv \frac{\partial\sigma_{ij}}{\partial t} + (\mathbf{v}\cdot\nabla)\sigma_{ij} - \left(\frac{\partial v_i}{\partial x_\alpha}\sigma_{\alpha j} + \frac{\partial v_j}{\partial x_\alpha}\sigma_{\alpha i}\right)$$

Here $\mathbf{v} = \{v_1, v_2, v_3\}$ is the fluid velocity, e_{ij} is the deformation rate tensor, σ_{ij} is the additional stress tensor (above the isotropic pressure p) and η, θ_1, θ_2—the viscosity, relaxation time and retardation (lag) time—are the constants of the fluid.

In the case of the inertialess steady flow of an incompressible fluid, the velocity vector turns out to be a two-dimensional vector, the pressure is independent of the transverse coordinate z and the velocity distribution is expressed in terms of the local pressure gradient using the Boussinesq formula, which is analogous to Darcy's law for plane motion

$$\mathbf{v} = -\frac{h^2}{2\eta}\left(1 - \frac{z^2}{h^2}\right)\nabla p, \quad v_z = 0$$

The stress field in a visco-elastic fluid then has the form

$$\sigma_{xx} = \frac{\theta_1 - \theta_2}{\eta} 2z^2 \left(\frac{\partial p}{\partial x}\right)^2, \ \sigma_{yy} = \frac{\theta_1 - \theta_2}{\eta} 2z^2 \left(\frac{\partial p}{\partial y}\right)^2, \ \sigma_{xy} = \frac{\theta_1 - \theta_2}{\eta} 2z^2 \frac{\partial p}{\partial x} \frac{\partial p}{\partial y}$$

$$\sigma_{xz} = \eta z \frac{\partial p}{\partial x}, \ \sigma_{yz} = \eta z \frac{\partial p}{\partial y}, \ \sigma_{zz} = 0$$

These relations are exact for flow between two unbounded plane plates in the case of a pressure gradient which is constant both in magnitude and direction.

We shall use the above formulae in the more general situation of a variable pressure gradient $\nabla p(x, y, t)$, assuming that all fields are slowly varying functions of the coordinates in the plane of the gap. After averaging over the width of the narrow gap between the plates, the flow will be described by the usual Boussinesq–Darcy relations for an incompressible fluid of viscosity η

$$\mathbf{u} \equiv \langle \mathbf{v} \rangle = -\frac{h^2}{3\eta} \nabla p, \ \Delta p = 0$$

and the field of the averaged stresses is simply determined from the pressure field

$$\langle \sigma_{xx} \rangle = c\left(\frac{\partial p}{\partial x}\right)^2, \ \langle \sigma_{yy} \rangle = c\left(\frac{\partial p}{\partial y}\right)^2, \ \langle \sigma_{xy} \rangle = c\left(\frac{\partial p}{\partial y}\right)\left(\frac{\partial p}{\partial x}\right), \ \langle \sigma_{zz} \rangle = 0$$

$$c = 2h^2(\theta_1 - \theta_2)/(3\eta)$$

It can be shown that these relations can be obtained within the framework of the formal procedure of expansion with respect to a small parameter which characterizes the slowness of the change of state with respect to the spatial variables. Although the flow in a bulk fluid in a similar "quasihomogeneous", "local" approximation is solely determined by the viscous properties, even small elastic stresses can have an influence on its stability through the boundary conditions.

We now consider the problem of the stability of a rectilinear, uniformly moving front for the displacement of a visco-elastic fluid by an inviscid fluid (a gas). In the case of the basic (unperturbed) flow in such a front $x = Ut$ and the boundary conditions for the velocity and the normal stress

$$u_x = -\frac{h^2}{3\eta} \frac{\partial p}{\partial x} = U, \ p - \langle \sigma_{xx} \rangle = -\frac{\alpha}{h} \tag{2.1}$$

are satisfied.

Here, the capillary pressure jump (α is the surface tension coefficient) is taken into account and, in addition, the curvature of the interfacial surface in a transverse direction to the plates is assumed to be constant and independent of the motion of the front which corresponds to ideal wetting of the plates of the trough which is filled with the fluid.

In the bulk of the fluid, we have

$$u_x = U, \ \frac{\partial p}{\partial x} = -\frac{3\eta}{h^2} U = \text{const}$$

For small perturbations of the front (1.5), the linearized boundary conditions for perturbations, denoted by a prime, in the perturbed front have the form

$$\frac{\partial \xi}{\partial t} = u'_x \Big|_{x=Ut}, \ \alpha \frac{\partial^2 \xi}{\partial y^2} = \left(\xi \frac{\partial p}{\partial x} + p' - \xi \frac{\partial \langle \sigma_{xx} \rangle}{\partial x} - \langle \sigma'_{xx} \rangle \right) \Big|_{x=Ut}$$

and, after using the relations for the perturbations of the flow in the bulk of the fluid

$$u'_x = -\frac{h^2}{3\eta} \frac{\partial p'}{\partial x}, \ \langle \sigma'_{xx} \rangle = -4(\theta_1 - \theta_2) U \frac{\partial p'}{\partial x}$$

they can be rewritten as conditions for perturbations of the pressure and the boundary

$$\frac{\partial\xi}{\partial t}=-\frac{h^2}{3\eta}\frac{\partial p'}{\partial x}\bigg|_{x=Ut},\quad \left(\frac{3\eta}{h^2}U+\alpha\frac{\partial^2}{\partial y^2}\right)\xi=\left[1+4(\theta_1-\theta_2)U\frac{\partial}{\partial x}\right]p'\bigg|_{x=Ut} \tag{2.2}$$

Pressure perturbations satisfy Laplace's equation. By virtue of the linearity of the problem for the perturbations, it is sufficient to consider perturbations which are periodic in x and t of the form

$$\xi(y,t)=A(t)\exp i(ky-\omega t),\quad p'=P(x-Ut)\exp i(ky-\omega t)$$

Then, the elementary pressure perturbations which decay on moving away from the front are given by the expression

$$p'\sim\exp[-|k|(x-Ut)+iky-i\omega t]$$

and, when this is substituted into boundary conditions (2.2), it leads to the following expression for the amplitude of the growth in the small sinusoidal perturbations of the boundary

$$A=A_0\exp\varkappa t;\quad \varkappa=\frac{1}{A}\frac{dA}{dt}=|k|(U-\frac{\alpha h^2}{3\eta}k^2)(1-4(\theta_1-\theta_2)U|k|)^{-1}$$

This expression differs from the classical result for the instability of a plane–parallel displacement front of a viscous fluid solely in the fact that there is an additional factor in the denominator which depends on the difference between the relaxation time and the retardation time of the fluid. The latter has no effect on the magnitude of the critical wavelength λ_c or the critical wave number.

$$k_c=2\pi/\lambda_c=(3\eta U/\alpha)^{1/2}/h$$

which correspond to the shortest-wavelength unstable perturbations. Only the increment in the growth of the perturbations changes and the rate of growth of unstable perturbations increases with wave numbers from the range

$$k\leqslant\min(k_c,k^0),\quad k^0=[4(\theta_1-\theta_2)U]^{-1}$$

When the fluid has a low elasticity (which corresponds to short relaxation and retardation times at a fixed viscosity), $k^0>k_c$ and all unstable perturbations are destabilized. In the opposite case, only the longest-wavelength perturbations are destabilized and, according to the formula obtained, stabilization occurs in the case of the shorter wavelength perturbations with $k^0<k<k_c$, which are unstable from the very outset, since the factor in the denominator changes sign. In general, stable and unstable perturbations formally change places when $k^0>k$. However, it should be remembered that all of the preceding derivation is based on the use of the asymptotic form of spatially slowly changing motions (that is, in the sense of long wavelength), so that it hardly follows to attribute any serious significance to the formal conclusion mentioned above.

The fastest growing perturbations with $k<k^0$ are characterized by a wave number

$$k_{\max}=\frac{k_c}{\sqrt{3}}\left[1-\frac{\sqrt{3}}{8}(\theta_1-\theta_2)Uk_c\right]^{-1/2}$$

that is, the wavelength of the fastest-growing perturbation becomes smaller under the influence of the elasticity of the fluid.

Development of the instability of a circular displacement front. Within the framework of the same "quasi-homogeneous", "local" approximation, for which the flow is described by the usual Darcy law and the visco-elastic effect of the normal stresses is only taken into consideration in the boundary conditions in the displacement front, we shall now consider the problem of the stability of the radial displacement of a visco-elastic fluid.

The velocity of the flow, averaged over the narrow gap between the plates and the stress field in the plane of the gap are described by the equations

$$\mathbf{u} = -\frac{h^2}{3\eta}\nabla p, \quad \Delta p = 0, \quad \langle \sigma_{ij} \rangle = c\frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j}$$

Using the boundary conditions in a circular displacement front $r = R(t)$

$$u_r\Big|_{r=R} = \frac{dR}{dt}, \quad (p - \langle \sigma_{rr} \rangle)\Big|_{r=R} = -\alpha\left(\frac{1}{h} + \frac{1}{R}\right)$$

which are an analogue of conditions (2.1) for the flow in the bulk of the fluid, we obtain

$$u_r = \frac{R}{r}\frac{dR}{dt}, \quad p = p_0 - \frac{3\eta}{h^2}R\frac{dR}{dt}\ln\frac{r}{R}$$

For small perturbations of the front (1.13), the linearized boundary conditions for the perturbations take the form

$$\left(u_r' + \frac{\partial u_r}{\partial r}\xi_m e^{im\varphi}\right)\Big|_{r=R} = \frac{d\xi_m}{dt}e^{im\varphi}$$

$$\left[p' - \langle \sigma_{rr}' \rangle + \frac{\partial}{\partial r}(p - \langle \sigma_{rr} \rangle)\xi_m e^{im\varphi}\right]_{r=R} = \frac{\alpha(1-m^2)}{R^2}\xi_m e^{im\varphi}$$

For perturbations in the bulk of the fluid, we find, when account is taken of the decay condition at infinity

$$p' = \frac{3\eta}{mh^2}ru_r' = \frac{3\eta}{mh^2}\left(\frac{R}{r}\right)^m \frac{d\xi_m R}{dt}e^{im\varphi}, \quad \langle \sigma_{rr}' \rangle = 4m(\theta_1 - \theta_2)\frac{d\ln R}{dt}p'$$

At the same time, the boundary conditions reduce to an equation for the increment in the growth of the small perturbations

$$\kappa_m = \frac{1}{\xi_m}\frac{d\xi_m}{dt}\left[1 - 4(\theta_1 - \theta_2)\frac{m}{R}\frac{dR}{dt}\right] = \frac{1-m}{R}\frac{dR}{dt} - \frac{\alpha h^2}{3\eta}\left(m^2 - 1\right)\frac{m}{R^3}$$

In the case when the change in the front radius with time follows a power law

$$R = qt^\gamma, \quad t = \tau R^\beta, \quad \beta = 1/\gamma$$

the equation for the growth of small perturbations of the front can be rewritten in the form

$$\kappa_m = \frac{1}{\xi_m}\frac{d\xi_m}{dR} = (m-1)\left[1 - \left(\frac{R_c}{R}\right)^{3-\beta}\right]\left[1 - \frac{m(\theta_1 - \theta_2)}{\beta\tau R^\beta}\right]^{-1}$$

$$R_c^{3-\beta} = \alpha\beta\tau m(m+1)h^2/(3\eta)$$

It is seen from this that, in the approximation being considered, the visco-elasticity, when

$$\Pi \equiv 4(\theta_1 - \theta_2)R^{-1}dR/dt \ll 1$$

has no effect on the critical radius R_c (the front is stable when $R < R_c$) and only somewhat increases the rate of growth of unstable perturbations (when $R > R_c$). Elasticity is found to have a greater effect on the higher harmonics (with larger m).

A pronounced acceleration of the development of instability and the replacement of a stability domain by an instability domain occur when the parameter Π formally tends to $1/m$ and on passing through these values.

The effect of the elastic parameter Π falls off rapidly as the radius of the front increases if $R(t)$ increases more slowly than exponentially.

Uniformly moving finger of an inviscid fluid. As the preceding analysis shows, the effect of the elasticity of a fluid on the behaviour of the free boundaries in Hele–Shaw flow reduces, in the first approximation, to a change in the boundary condition for the pressure on the free boundary which (when there are no capillary forces) takes the form

$$-p+c\left(\frac{\partial p}{\partial n}\right)^2=0$$

where n is the direction of the normal to the boundary. In the bulk of the fluid, the equations of motion in the first approximation remain the same as for a Newtonian viscous fluid and reduce to Laplace's equation in the case of the velocity potential

$$\Delta\varphi=0,\ \ \varphi=-h^2p/(3\eta)$$

This enables one to advance to some extent when investigating more complex, non-one-dimensional motions with a free boundary. In particular, we shall consider the classical Saffman–Taylor problem on the steady motion of a finger of an inviscid fluid along the axis of a strip-shaped Hele–Shaw cell (Fig. 1a) filled with a visco-elastic fluid. Within the framework of the approximation which has been adopted, the problem reduces to finding a pair of conjugate harmonic functions φ and ψ in the domain $ABCDEA$ (Fig. 1) which satisfy the conditions

$$\psi=0,(x,y)\in AB;\ \ \psi=\lambda UH,\ \ (x,y)\in CD$$

$$\varphi=-\beta\left(\frac{\partial\varphi}{\partial n}\right)^2,\ (x,y)\in EA;\ \beta=\frac{3\eta}{h^2}c=2(\theta_1-\theta_2)$$

Here, the finger boundary EA itself is to be determined; the corresponding additional boundary condition expresses the fact that the finger moves as a solid body at a speed U, and we therefore have that $\psi = Uy$ along EA.

This problem can be significantly simplified and made more obvious by introducing [10] the auxiliary analytic function

$$\chi=\xi+i\eta=W-Uz,\ \ W=\varphi+i\psi,\ \ z=x+iy \tag{2.3}$$

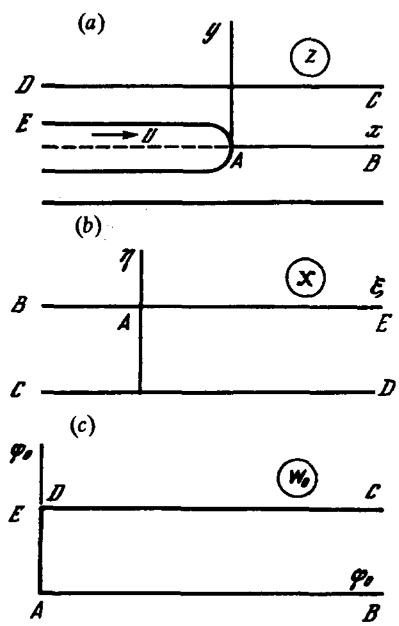

Fig. 1.

The strip Π_χ which corresponds to the points shown in Fig. 1(b) is the image representation of the unknown flow domain in the χ plane; the complex coordinate z and the complex potential W are analytic functions of χ in Π_χ which satisfy the boundary conditions

$$\psi = \operatorname{Im} W = 0,\ y = \operatorname{Im} z = 0,\ \eta = 0,\ \xi < 0$$

$$\psi = \operatorname{Im} W = \lambda UH,\ y = \operatorname{Im} z = H,\ \eta = -(1-\lambda)UH,\ -\infty < \xi < \infty \tag{2.4}$$

$$\varphi + \beta(\partial\psi / \partial s)^2 = 0,\ ds / d\xi = |dz / d\xi|,\ \eta = 0,\ \xi > 0$$

and are related by (2.3).

In the case of small β, for which the problem being considered really makes sense, it is easy to find an approximate solution which is iteratively constructed. On putting $\beta = 0$ in expressions (2.3) and (2.4), we first obtain the zeroth approximation problem

$$\psi_0 = 0,\ \eta = 0,\ \xi < 0;\ \psi_0 = \lambda UH,\ \eta = -(1-\lambda)UH,\ -\infty < \xi < \infty$$

$$\varphi_0 = 0,\ \eta = 0,\ \xi > 0,\ z_0 = (W_0 - \chi)/U$$

The problems of finding $W_0(\chi)$ and $z_0(\chi)$ are now separated and can be solved successively.

The solution of the zeroth approximation problem has been found in [3], and the result has the form

$$\frac{z}{H} = \frac{W_0}{\lambda UH} + \frac{2}{\pi}(1-\lambda)\ln\left[\frac{1}{2}[1+\exp(-\mu W_0)]\right],\ \mu \equiv \frac{\pi}{\lambda UH}$$

$$\chi = -(1-\lambda)UH\left[\frac{W_0}{\lambda UH} + \frac{2}{\pi}\ln\frac{1}{2}[1+\exp(-\mu W_0)]\right] \tag{2.5}$$

Here, $W_0 = \varphi_0 + i\psi_0$ is the complex potential in the zeroth approximation, which varies in the half-strip $0 \leqslant \varphi_0 < \infty$, $0 \leqslant \psi_0 \leqslant \lambda UH$ shown in Fig. 1(c). It is henceforth more convenient to use the initial potential W_0, which is uniquely related to χ by the second relation of (2.5), as an argument when constructing the successive approximations. For them, we have the following sequence of problems (in the χ plane)

$$\psi_k = 0,\ \eta = 0,\ \xi < 0;\ \psi_k = \lambda HU,\ \eta = -(1-\lambda)HU,\ -\infty < \xi < \infty$$

$$\varphi_k = -\beta\left[\frac{\partial\psi_{k-1}}{\partial\xi}\left|\frac{dz_{k-1}}{d\xi}\right|^{-1}\right]^2 \equiv \varphi_k(\xi),\ \eta = 0,\ \xi > 0;\ z_k = \frac{(W_k - \chi)}{U} \tag{2.6}$$

$$k = 1,\dots,n,\dots$$

For a known function $\varphi_k(\xi)$, the mixed boundary-value problem (2.6) is solved in closed form using a standard method. First, using the transformation

$$\zeta = \tau + i\theta = \exp\left[-\frac{\pi\chi}{(1-\lambda)UH}\right] \tag{2.7}$$

the domain of variation of the independent variable is mapped into the upper half-plane, $\theta > 0$, and the mixed boundary-value problem which arises is solved using the Keldysh–Sedov formula. However, we shall use a representation of all the functions in terms of ψ_0 and φ_0 which are treated as coordinates in the auxiliary plane W_0. Here ψ_k and φ_k are harmonic functions of ψ_0 and φ_0.

From (2.6), we have for the first approximation ($k = 1$)

$$\varphi_1\Big|_{\varphi_0=0} = -\beta\left[\frac{\partial\psi_0}{\partial\xi}\left|\frac{dz_0}{d\xi}\right|^{-1}\right]^2 \equiv -\beta\left|\frac{dz_0}{dW_0}\right|^{-2}_{\varphi_0=0}$$

It follows from (2.5) that

$$\frac{dz_0}{dW_0}\bigg|_{\varphi_0=0} = \frac{i}{\lambda U}\left[1-(1-\lambda)\exp\left[-\frac{i\mu\psi_0}{2}\right]\left(\cos\frac{\mu\psi_0}{2}\right)^{-1}\right]$$
$$\varphi_1\big|_{\varphi_0=0} = -\beta\lambda^2U^2\left[\lambda^2+(1-\lambda)^2\,\mathrm{tg}^2\,\frac{\mu\psi_0}{2}\right]^{-1} \tag{2.8}$$

Relation (2.8) fixes the value of the required function on the left-hand boundary ($\varphi_0 = 0$) of the half-strip in Fig. 1(c). In addition, it must satisfy the conditions

$$\partial\varphi_1/\partial\psi_0 = 0,\ \ \psi_0 = 0;\ \ \psi_0 = \lambda UH,\ \ \varphi_0 \geqslant 0;\ \ \varphi_1 \sim \varphi_0,\ \ \varphi_0 \to \infty$$

Solving Laplace's equation for φ_1 with the above-mentioned boundary conditions, we find

$$\varphi_1 = \varphi_0 + \sum_{m=0}^{\infty} C_m \cos m\mu\psi_0 \exp(-m\mu\varphi_0)$$

$$\psi_1 = \psi_0 - \sum_{m=0}^{\infty} C_m \sin m\mu\psi_0 \exp(-m\mu\varphi_0) \tag{2.9}$$
$$W_1 = W_0 + \sum_{m=0}^{\infty} C_m \exp(-m\mu W_0)$$

$$z_1 = \frac{W_1-\chi}{U} = z_0(W_0) + \frac{1}{U}\sum_{m=0}^{\infty} C_m \exp(-m\mu W_0) \tag{2.10}$$

$$C_m = -2\beta\lambda UH^{-1}\int_0^{\lambda UH} \cos m\mu\psi_0\left[\lambda^2+(1-\lambda)^2\,\mathrm{tg}^2\,\frac{\mu\psi_0}{2}\right]^{-1} d\psi_0 \tag{2.11}$$

Relations (2.9)–(2.11) give a complete solution of the first approximation of the problem for arbitrary λ. As in the Saffman–Taylor problem, the magnitude of λ, the asymptotic value of the width of the finger, remains indefinite and a solution of the problem exists for all λ between 0 and 1. The unknown boundary of the finger is obtained if one puts in (2.10)

$$W_0 = i\psi_0,\ \ 0 \leqslant \psi_0 \leqslant \lambda UH$$

$$z_1 = \frac{1}{\lambda U}W_0 + \frac{2}{\pi}H(1-\lambda)\ln\frac{1}{2}[1+\exp(-\mu\psi_0)] + \frac{1}{U}\sum_{m=0}^{\infty} C_m \exp(-im\mu\psi_0)$$

The result turns out to be particularly simple when $\lambda = 1/2$ (which corresponds to the stable fingers observed experimentally at sufficiently high rates of displacement). The integrand in (2.11) then becomes $2(1 + \cos\mu\psi_0)\cos m\mu\psi_0$. Only the first ($m = 1$) Fourier coefficient is therefore non-zero in this case, $C_m = 0, m \neq 0$; $C_1 = -2\beta\lambda^2U^2$. Hence, for the perturbed boundary of the finger, we obtain

$$z_1(\psi_0) = z_0(\psi_0) - 2\beta\lambda^2U^2[\cos\frac{2\pi\psi_0}{UH} - i\sin\frac{2\pi\psi_0}{UH})$$

In the rear part of the finger ($\psi_0 \to UH/2$), the points of the boundary are displaced in the forward

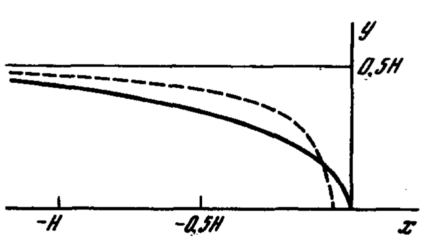

Fig. 2.

direction relative to the unperturbed position, since $\cos[2\pi\psi_0/(UH)] \to -1$ while, in the leading part ($\psi_0 \to 0$), they are displaced backwards by the same amount $\beta U/2$ (the finger shape is, of course, only determined with an accuracy up to a translation as a rigid whole along the axis of the channel). Hence, the transitional zone contracts and the finger becomes blunter. The unperturbed and perturbed shapes of the finger are shown in Fig. 2 for $\gamma = \beta U = 0.2$.

Note that the direct physical interpretation of this result as it applies to fingers which develop during the unstable displacement of elastic fluids is made difficult by the fact that the initial problem has a continuous spectrum of permitted steady shapes of the fingers (for any λ) and it is unclear how taking elasticity into account affects the choice of the "correct" (stable) solution from this spectrum. The problem of choice has so far only been investigated taking account of the effect of capillary forces on the finger free boundary.

In conclusion, we note that the formal mathematical investigation carried out in this paper essentially refers to any flow in a Hele–Shaw trough or to its analogues of another physical nature when the pressure distribution in the bulk of the fluid is described by Laplace's equation and the bounding value of the pressure at the free boundary is a function of the rate of advance of the boundary

$$p|_\Gamma = f(V_n)|_\Gamma$$

This relation may be a consequence not only of the visco-elastic behaviour of the fluid but also of other physical effects such as, for example, the non-constancy of the dynamic boundary wetting angle of the fluid being displaced. Corresponding effects for Hele–Shaw flows have been studied previously in [12–14]. We do not rule out the possibility that it is they, in fact, and not the contribution to the capillary pressure from the curvature of the front in the plane of the cell which is conveniently taken into account, which play a fundamental part in the dynamics of the boundaries.

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REFERENCES

1. Wilson, S. D. R., The Taylor–Saffman problem for a non-Newtonian liquid. *J. Fluid Mech.*, 1990, **220**, 413–425.
2. Homsy, G. M., Viscous fingering in porous media. *Ann. Rev. Fluid Mech.*, 1987, **19**, 271–311.
3. Saffman, P. G. and Taylor, G., The penetration of a fluid into a porous medium or Hele–Shaw cell containing a more viscous liquid. *Proc. R. Soc. London Ser. A*, 1958, **245**(1242), 312–329, 1958.
4. Chuoke, R. L., Neurs, P. and van der Poel, C., The instability of slow, immiscible, viscous liquid–liquid displacements in permeable media. *Petrol. Trans. AIME*, 1959, **216**, 188–194; Summary: *J. Petrol. Technol.*, 1959, **11**(7), 64, 73.
5. Bernadiner, M. G. and Yentov, V. M., *Hydrodynamic Theory of the Filtration of Anomalous Fluids*. Nauka, Moscow, 1975.
6. Pascal, H., Stability of moving interface in a porous medium for non-Newtonian displacing fluids and its application in oil displacement mechanism. *Acta Mech.*, 1986, **58**(1–2), 81–91.
7. Pascal, H., Rheological effects of non-Newtonian behavior of displacing fluids on stability of a moving interface in radial oil displacement mechanism. *Int. J. Engng Sci.*, 1986, **24**(9), 1465–1476.
8. Pascal, H., Stability of non-Newtonian fluid interface in a porous medium and its applications in an oil displacement mechanism. *J. Colloid Interface Sci.*, 1988, **123**(1), 14–23.
9. Sader, J. E., Chan, D. J. C. and Haghes, B. D., Non-Newtonian effects on immiscible viscous fingering in a radial Hele–Shaw cell, *Phys Rev. Ser. E*, 1994, **49**(1), 420–432.
10. Taylor, G. L. and Saffman, P. G., A note on the motion of bubbles in a Hele–Shaw cell and porous medium. *Q. J. Mech. Appl. Math.*, 1959, **12**(3), 265–279.
11. Lavrent'yev, M. A. and Shabat, B. V., *Methods of the Theory of Functions of a Complex Variable*. Fizmatgiz, Moscow, 1958.
12. Park, C.-W. and Homsy, G. M., Two phase displacement in Hele–Shaw cells: theory. *J. Fluid Mech.*, 1984, **139**, 291–308.
13. Reinelt, D. A., Interface conditions for two-phase displacement in Hele–Shaw cells. *J. Fluid Mech.*, 1987, **183**, 219–234.
14. Ro, J. S. and Homsy, G. M., Viscoelastic free surface flows; thin film hydrodynamics of Hele–Shaw and dip coating flows. *J. Non-newt. Fluid Mech.*, 1995, **57**(2), 203–225.

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